

ON MODULO AG-GROUPOIDS

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ABSTRACT. A groupoid G is called an AG-groupoid if it satisfies the left invertive law: $(ab)c = (cb)a$. An AG-group G , is an AG-groupoid with left identity $e \in G$ (that is, $ea = a$ for all $a \in G$) and for all $a \in G$ there exists $a^{-1} \in G$ such that $a^{-1}a = aa^{-1} = e$. In this article we introduce the concept of AG-groupoids (mod n) and AG-group (mod n) using Vasantha's constructions [1]. This enables us to prove that AG-groupoids (mod n) and AG-groups (mod n) exist for every integer $n \geq 3$. We also give some nice characterizations of some classes of AG-groupoids in terms of AG-groupoids (mod n).

1. INTRODUCTION

Construction for any algebraic structure is always very important for its developement. The examples so obtained are sometimes not even possible through computers to come by. Open problems and conjunctures are often answered by constructing examples for them. Several construction are available for forming quasigroup and loops. For example, an infinite family of nonassociative noncommutative C-loops whose smallest member is the smallest non-associative noncommutative C-loop of order 12 has been constructed in [2]. We can obtained manually a C-loop of this family of much much higher order that might not be possible through computers. Sometimes a construction can be implemented in computer which then makes the job easier. For example, a construction of AG-groups from abelian groups has been implemented in GAP, the details of this construction and its implementation can be found in [3]. M. S. Kamran has also discussed another construction of AG-groups from abelian groups in his PhD thesis [5]. Several types of a structure can be obtained from each other through some specific constructions. For example, this has been done for AG-groupoids [4]. Several constructions of groupoids have been done by W. B. Vasantha [1]. In this paper we extend Vasantha's constructions to AG-groupoids by imposing some conditions on them. The structure of AG-groupoid is considered one of the most interesting structure among the non-associative structures. A considerable achivement has been done for the improvement of AG-groupoids by various researchers see for instance[11, 12, 13, 14, 15]. This extension will really give another push to the study of AG-groupoids which gets broadened quite rapidly these days. This will also make Vasantha's constructions of groupoids more valuable. The paper can be considered as a sort of applications of the mentioned constructions. It also

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gives some nice characterizations of some classes of AG-groupoids in new scenario, that is, in terms of AG-groupoids (mod n).

In Section 2 we will establish existence of AG-groupoid (mod n) and in Section 3 we will construct AG-groups (mod n) and will provide some finite examples to show their existence.

An AG-groupoid (or LA-semigroup) G is a groupoid in which the left invertive law: $(ab)c = (cb)a$ holds [6]. An AG-groupoid is a generalization of a commutative semigroup and an AG-group is a generalization of abelian group. Recently some new classes of AG-groupoids have been discovered in [8] and [10]. However we will need the following definitions.

An AG-groupoid G is called :

- (i) an AG-band, if $a^2 = a \forall a \in G$ [7];
- (ii) an AG-group, if $b * a = c * a \Rightarrow a * b = a * c$;
- (iii) a T_l^3 -AG-groupoid, if $a * b = a * c \Rightarrow b * a = c * a$;
- (iv) a T_r^3 -AG-groupoid, if $b * a = c * a \Rightarrow a * b = a * c$;
- (v) a transitively commutative AG-groupoid, if $a * b = b * a$ and $b * c = c * b \Rightarrow a * c = c * a$;
- (vi) a cancellative AG-groupoid, if $a * x = a * y \Rightarrow x = y$.

The following definitions are needed from [1].

Definition 1. Let $Z_n = \{0, 1, 2, \dots, n-1\}; n \geq 3$. For $a, b \in Z_n \setminus \{0\}$, define a binary operation $*$ on Z_n as follows $a * b = ta + ub \pmod{n}$ where t, u are two distinct elements in $Z_n \setminus \{0\}$ and $(t, u) = 1$ here ‘ $+$ ’ is the usual addition of two integers and ‘ ta ’ means the product of the two integers ‘ t ’ and ‘ a ’. This groupoid will be denoted by $(Z_n, (t, u), *)$ or in short by $Z_n(t, u)$. By varying $t, u \in Z_n \setminus \{0\}$ with $(t, u) = 1$ we get a collection of groupoids for a fixed integer n . This collection of groupoids is denoted by $Z(n)$ that is $Z(n) = \{(Z_n, (t, u), *) \mid \text{for distinct integers } t, u \in Z_n \setminus \{0\} \text{ such that } (t, u) = 1\}$. Clearly every groupoid in this class is of order n .

Definition 2. If (t, u) need not always be relative prime but $t \neq u$ and $t, u \in Z_n \setminus \{0\}$ in Definition 1 we get a new extended class of $Z(n)$ denoted by $Z^*(n)$.

Definition 3. If (t, u) need not always be distinct in Definition 1 we get a new enlarge class of $Z^*(n)$ denoted by $Z^{**}(n)$.

Definition 4. If $t, u \in Z_n$ where t or u can also be zero in Definition 1 we get a new class denoted by $Z^{***}(n)$ contains $Z^{**}(n)$.

2. EXISTENCE OF AG-GROUPOIDS (MOD n)

In this section, we introduce AG-groupoids (mod n) as a subclass of the class $Z^{***}(n)$. We study these AG-groupoids (mod n) and obtain some results about them. The following theorem guarantees the existence of AG-groupoids (mod n) for $n \geq 3$, indeed it provides us with a simple way of construction of AG-groupoids (mod n) of any finite order.

Theorem 1. Let $Z_n = \{0, 1, 2, \dots, n-1\}$, $n \geq 3$, $n < \infty$. A groupoid in $Z^{***}(n)$ is an AG-groupoid if $t^2 \cong u \pmod{n}$ for any $t, u \in Z_n$.

Proof. Let $Z_n = \{0, 1, 2, \dots, n-1\}$, $n \geq 3$, $n < \infty$; satisfies $t^2 \cong u \pmod{n}$ for any $t, u \in Z_n \setminus \{0\}$. To show $Z_n(t, u)$ is an AG-groupoid, we have to show that the left invertive law, that is, $(a \cdot b) \cdot c = (c \cdot b) \cdot a \ \forall a, b, c \in Z_n$ holds. Now

$$\begin{aligned} (a \cdot b) \cdot c &\cong (t(ta + ub) + uc) \pmod{n} \\ &\cong (t^2a + tub + uc) \pmod{n} \end{aligned}$$

and

$$\begin{aligned} (c \cdot b) \cdot a &\cong (t(tc + ub) + ua) \pmod{n} \\ &\cong (t^2c + tub + ua) \pmod{n} \end{aligned}$$

By using hypothesis, we get $(a \cdot b) \cdot c = (c \cdot b) \cdot a \ \forall a, b, c \in Z_n$.

Now we show that the class (Z_n, \cdot) is nonassociative in general:

$$\begin{aligned} (a \cdot b) \cdot c &\cong (t^2a + tub + uc) \pmod{n}, \\ a \cdot (b \cdot c) &\cong (ta + utb + u^2c) \pmod{n}. \end{aligned}$$

Since this is not necessary that

$$t^2a + tub + uc \cong (ta + utb + u^2c) \pmod{n}.$$

Hence $Z_n(t, u)$ is an AG-groupoid which may or may not be associative. ■

We denote this AG-groupoid by $\{Z_n, (t, u), \cdot\}$ -AG-groupoid (mod n) or in short by $Z_n(t, u)$ -AG-groupoid (mod n). For varying values of t and u and by putting some conditions on t and u , we get different classes of AG-groupoids (mod n) for some fixed integer $n \geq 3$. These new classes of AG-groupoids (mod n) will be denoted by $Z_{AG}^*(n)$, $Z_{AG}^{**}(n)$ and $Z_{AG}^{***}(n)$. In the following we list some examples to show the existence of these modulo AG-groupoids.

- (i) $Z_3(2, 1)$ in $Z(3)$ is an AG-groupoid (in fact an AG-group):

\cdot	0	1	2
0	0	1	2
1	2	0	1
2	1	2	0

(ii) $Z_8(6, 4)$ in $Z^*(8)$ is an AG-groupoid:

\cdot	0	1	2	3	4	5	6	7
0	0	4	0	4	0	4	0	4
1	6	2	6	2	6	2	6	2
2	4	0	4	0	4	0	4	0
3	2	6	2	6	2	6	2	6
4	0	4	0	4	0	4	0	4
5	6	2	6	2	6	2	6	2
6	4	0	4	0	4	0	4	0
7	2	6	2	6	2	6	2	6

(iii) $Z_{AG}^{**}(3) = \{Z_3(1, 1), Z_3(2, 1)\}$.

(iv) $Z_{AG}^{***}(4) = \{Z_4(1, 1), Z_4(2, 0), Z_4(3, 1)\}$.

(v) $Z_{AG}^{**}(5) = \{Z_5(1, 1), Z_5(4, 1), Z_5(3, 4), Z_5(2, 4)\}$.

(vi) $Z_{AG}^{**}(6) = \{Z_6(1, 1), Z_6(2, 4), Z_6(3, 3), Z_6(4, 4), Z_6(5, 1)\}$ and so on.

We immediately have the following consequences of Theorem 1.

Corollary 1. Any AG-groupoid in $Z_{AG}^{***}(n)$ is a commutative semigroup if $t = u$.

Proof. If $t = u$ then the binary operation becomes commutative which forces associativity in the AG-groupoid. ■

Example 1. $Z_6(4, 4)$ in $Z_{AG}^{**}(6)$ is a commutative semigroup given by the table:

\cdot	0	1	2	3	4	5
0	0	4	2	0	4	2
1	4	2	0	4	2	0
2	2	0	4	2	0	4
3	0	4	2	0	4	2
4	4	2	0	4	2	0
5	2	0	4	2	0	4

Next we characterize AG-groupoids (mod n).

Theorem 2. An AG-groupoid in $Z_{AG}^{**}(n)$ is a T^3 -AG-groupoid, if $t = u$.

Proof. Let $t = u$, then to show that an AG-groupoid in $Z_{AG}^{**}(n)$ is a T^3 -AG-groupoid, we will have to show that it is a T_l^3 -AG-groupoid as well as a T_r^3 -AG-groupoid.

For T_l^3 -AG-groupoid, let

$$\begin{aligned} a \cdot b &= a \cdot c \\ \Rightarrow (ta + ub) &\cong (ta + uc)(\text{mod } n) \\ \Rightarrow ub &\cong uc(\text{mod } n) \\ \Rightarrow tb &\cong tc(\text{mod } n) \end{aligned} \quad (2.1)$$

Now

$$\begin{aligned} b \cdot a &\cong (tb + ua)(\text{mod } n) \\ &\cong (tc + ua)(\text{mod } n) \quad (\text{by Equation 2.1}) \\ \Rightarrow b \cdot a &= c \cdot a \end{aligned}$$

Hence an AG-groupoid in $Z_{AG}^{**}(n)$ is T_l^3 -AG-groupoid. Similarly we can show that an AG-groupoid in $Z_{AG}^{**}(n)$ is T_r^3 -AG-groupoid. Hence any AG-groupoid in $Z_{AG}^{**}(n)$ is T^3 -AG-groupoid if $t = u$. ■

Theorem 3. Every AG-groupoid in $Z_{AG}^*(n)$ is a T^3 -AG-groupoid, if n is prime.

Proof. Let n be any prime number then to show that an AG-groupoid in $Z_{AG}^*(n)$ is a T^3 -AG-groupoid, we will have to show that it is a T_l^3 -AG-groupoid as well as a T_r^3 -AG-groupoid.

For T_l^3 -AG-groupoid, let $a, b, c \in G$, and

$$\begin{aligned} a \cdot b &= a \cdot c \\ \Rightarrow (ta + ub) &\cong (ta + uc)(\text{mod } n) \\ \Rightarrow ub &\cong uc(\text{mod } n) \\ \Rightarrow u(b - c) &\cong 0(\text{mod } n) \end{aligned}$$

as $n \nmid u$, because n is a prime number. Therefore, $n \mid (b - c) \Rightarrow b \cong c(\text{mod } n)$, and consequently;

$$\begin{aligned} b \cdot a &\cong (tb + ua)(\text{mod } n) \\ &\cong (tc + ua)(\text{mod } n) \\ \Rightarrow b \cdot a &= c \cdot a \end{aligned}$$

Hence every AG-groupoid G in $Z_{AG}^*(n)$ is T_l^3 -AG-groupoid. Similarly we can show that every AG-groupoid in $Z_{AG}^*(n)$ is T_r^3 -AG-groupoid. Hence every AG-groupoid in $Z_{AG}^*(n)$ is T^3 -AG-groupoid if n is prime. ■

Example 2. $Z_5(3, 4)$ in $Z_{AG}^*(5)$ is a T^3 -AG-groupoid:

\cdot	0	1	2	3	4
0	0	4	3	2	1
1	3	2	1	0	4
2	1	0	4	3	2
3	4	3	2	1	0
4	2	1	0	4	3

Also in Example 1; $Z_6(4, 4)$ in $Z_{AG}^*(6)$ is a T^3 -AG-groupoid. However, the result is not true in general. For example, $Z_8(6, 4)$ is not a T^3 -AG-groupoid:

\cdot	0	1	2	3	4	5	6	7
0	0	4	0	4	0	4	0	4
1	6	2	6	2	6	2	6	2
2	4	0	4	0	4	0	4	0
3	2	6	2	6	2	6	2	6
4	0	4	0	4	0	4	0	4
5	6	2	6	2	6	2	6	2
6	4	0	4	0	4	0	4	0
7	2	6	2	6	2	6	2	6

The following theorem shows that $Z_{AG}^*(n)$ is a subclass of transitively commutative AG-groupoid.

Theorem 4. Every AG-groupoid in $Z_{AG}^*(n)$ is transitively commutative AG-groupoid.

Proof. To show that every AG-groupoids in $Z_{AG}^*(n)$, is transitively commutative AG-groupoid it is sufficient if we show that an arbitrary AG-groupoid H in $Z_{AG}^*(n)$ is transitively commutative AG-groupoid. Now for any $a, b, c \in H$, we show that for $a \cdot b = b \cdot a$ and $b \cdot c = c \cdot b \Rightarrow a \cdot c = c \cdot a$. Let

$$\begin{aligned}
 a \cdot b &= b \cdot a \\
 \Rightarrow (ta + ub) &\cong (tb + ua)(\text{mod } n) \\
 \Rightarrow t(a - b) + u(b - a) &\cong 0(\text{mod } n)
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 b \cdot c &= c \cdot b \\
 \Rightarrow (tb + uc) &\cong (tc + ub)(\text{mod } n) \\
 \Rightarrow t(b - c) + u(c - b) &\cong 0(\text{mod } n)
 \end{aligned}$$

$$\begin{aligned}
& \text{as } n \mid t(a-b) + u(b-a) \text{ and } n \mid t(b-c) + u(c-b) \Rightarrow n \mid t(a-b) + u(b-a) + t(b-c) + u(c-b) \\
& \Rightarrow t(a-b) + u(b-a) + t(b-c) + u(c-b) \cong 0(\text{mod } n) \\
& \quad t(a-c) + u(-a+c) \cong 0(\text{mod } n) \\
& \quad (ta + uc) - (tc + ua) \cong 0(\text{mod } n) \\
& \quad \quad ta + uc \cong (tc + ua)(\text{mod } n) \\
& \quad \quad \Rightarrow ac = ca.
\end{aligned}$$

Hence every AG-groupoids in $Z_{AG}^*(n)$ is transitively commutative AG-groupoid. ■

Example 3. $Z_7(5, 4)$ is transitively commutative AG-groupoid:

\cdot	0	1	2	3	4	5	6
0	0	4	1	5	2	6	3
1	5	2	6	3	0	4	1
2	3	0	4	1	5	2	6
3	1	5	2	6	3	0	4
4	6	3	0	4	1	5	2
5	4	1	5	2	6	3	0
6	2	6	3	0	4	1	5

Theorem 5. Every AG-groupoid in $Z_{AG}^*(n)$ is a cancellative AG-groupoid, if n is prime.

Proof. To show that for any prime number n ; every AG-groupoid in $Z_{AG}^*(n)$ is a cancellative AG-groupoid, it is sufficient if we show an arbitrary AG-groupoid is left cancellative AG-groupoid.

For left cancellative AG-groupoid, let

$$\begin{aligned}
a \cdot x &= a \cdot y \\
\Rightarrow (ta + ux) &\cong (ta + uy)(\text{mod } n) \\
\Rightarrow u(x - y) &\cong 0(\text{mod } n)
\end{aligned}$$

as $n \nmid u$, because n is a prime number. Therefore, it means that $n \mid (x - y) \Rightarrow x \cong y(\text{mod } n)$. Hence every AG-groupoid is left cancellative. As every left cancellative AG-groupoid is right cancellative AG-groupoid [10]. Hence every AG-groupoid in $Z_{AG}^*(n)$ is a cancellative AG-groupoid. ■

Example 4. $Z_5(3, 4)$ in $Z_{AG}^*(5)$ is a cancellative AG-groupoid:

\cdot	0	1	2	3	4
0	0	4	3	2	1
1	3	2	1	0	4
2	1	0	4	3	2
3	4	3	2	1	0
4	2	1	0	4	3

Theorem 6. An AG-groupoid in $Z_{AG}^{***}(n)$ is an AG-band if $t + u = 1$.

Proof. Let $t + u = 1$, to show that $Z_{AG}^{***}(n)$ is an AG-band it is sufficient to show that $a \cdot a \cong a(\text{mod } n)$;

$$\begin{aligned}
 a \cdot a &\cong (ta + ua)(\text{mod } n) \\
 &\cong a(t + u)(\text{mod } n) \\
 &\cong a(\text{mod } n)
 \end{aligned}$$

Hence the claim. ■

Example 5. $Z_5(2, 4)$ is an AG-band:

\cdot	0	1	2	3	4
0	0	4	3	2	1
1	2	1	0	4	3
2	4	3	2	1	0
3	1	0	4	3	2
4	3	2	1	0	4

3. EXISTENCE OF AG-GROUPS (MOD n)

In this section, we introduce a special class of AG-groupoids (mod n), namely AG-groups (mod n) and give some of its characterizations. The following theorem shows the existence of AG-groups (mod n) for $n \geq 3$, and indeed it gives a simple way of construction of AG-groups of any finite order.

Theorem 7. Let $Z_n = \{0, 1, 2, \dots, n-1\}$ $n \geq 3$, $n < \infty$. A groupoid in $Z^{**}(n)$ is an AG-group if $t^2 \cong 1(\text{mod } n)$ for $t \in Z_n \setminus \{0\}$.

Proof. Given that $Z_n = \{0, 1, 2, \dots, n-1\}$ $n \geq 3$, $n < \infty$; satisfies $t^2 \cong 1(\text{mod } n)$ for $t \in Z_n \setminus \{0\}$, we have to show $Z_n(t, 1)$ in $Z^{**}(n)$ is an AG-group.

We show that the left invertive law, $(a \cdot b) \cdot c = (c \cdot b) \cdot a$ holds in Z_n ;

$$\begin{aligned} (a \cdot b) \cdot c &\cong (t(ta + b) + c)(\text{mod } n) \\ &\cong (t^2a + tb + c)(\text{mod } n) \end{aligned}$$

and

$$\begin{aligned} (c \cdot b) \cdot a &\cong (t(tc + b) + a)(\text{mod } n) \\ &\cong (t^2c + tb + a)(\text{mod } n) \end{aligned}$$

Hence $Z_n(t, 1)$ is an AG-groupoid as $t^2 \cong 1(\text{mod } n)$ and $(a \cdot b) \cdot c = (c \cdot b) \cdot a$.

Existence of left identity: '0' is the correspondent left identity;

$$0 \cdot x \cong x(\text{mod } n) \forall x \in Z_n$$

but

$$x \cdot 0 \cong (tx)(\text{mod } n).$$

Existence of inverses: $(n - 1)tx = -tx$ is the inverse of $x \forall x \in Z_n$;

$$\begin{aligned} (-tx) \cdot x &\cong (t(-tx) + x)(\text{mod } n) \\ &\cong (-(t^2 - 1)x)(\text{mod } n) \\ &\cong 0(\text{mod } n) \end{aligned}$$

and

$$\begin{aligned} x \cdot (-tx) &\cong (tx + (-tx))(\text{mod } n) \\ &\cong 0(\text{mod } n) \end{aligned}$$

Hence (Z_n, \cdot) is an AG-group $(\text{mod } n)$. ■

We denote this AG-group $(\text{mod } n)$ by $\{Z_n, (t, 1), \cdot\}$ -AG-group $(\text{mod } n)$ or in short by $Z_n(t, 1)$ -AG-group $(\text{mod } n)$.

Corollary 2. *Any AG-groupoid in $Z_{AG}^{**}(n)$ is an abelian group if $t = 1$.*

Proof. If $t = 1$ then 1 becomes the identity of the $Z_n(1, 1)$ -AG-group $(\text{mod } n)$ and so it becomes abelian group by [10, Theorem 2]. ■

Corollary 3. *Let $Z_n = \{0, 1, 2, \dots, n-1\}$, $n \geq 3$, $n < \infty$. Then $Z_n(n-1, 1)$ is an AG-group $(\text{mod } n)$.*

Proof. Since $(n-1)^2 \cong 1 \pmod{n}$. The proof now follows by Theorem 7. ■

We denote this AG-group by $\{Z_n, (n-1, 1), \cdot\}$ -AG-group \pmod{n} or in short by $Z_n(n-1, 1)$ -AG-group \pmod{n} .

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